

A Strong Edge-Coloring of Graphs with Maximum Degree 4 Using 22 Colors

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Abstract

In 1985, Erdős and Nešetřil conjectured that the strong edge-coloring number of a graph is bounded above by $\frac{5}{4}\Delta^2$ when Δ is even and $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ when Δ is odd. They gave a simple construction which requires this many colors. The conjecture has been verified for $\Delta \leq 3$. For $\Delta = 4$, the conjectured bound is 20. Previously, the best known upper bound was 23 due to Horak. In this paper we give an algorithm that uses at most 22 colors.

1 Introduction

A *proper edge-coloring* is an assignment of a color to each edge of a graph so that no two edges with a common endpoint receive the same color. A *strong edge-coloring* is a proper edge-coloring, with the further condition that no two edges with the same color lie on a path of length three. The *strong edge chromatic number* is the minimum number of colors that allow a strong edge-coloring. In this paper we consider the maximum possible strong edge chromatic number as a function of the maximum degree of the graph. For other variations of the problem, we refer the reader to a brief survey by West [6] and a paper by Faudree, Schelp, Gyárfás and Tuza [3].

We use Δ to denote the maximum degree of the graph. In 1985 Erdős and Nešetřil conjectured that the strong edge chromatic number of a graph is at most $\frac{5}{4}\Delta^2$ for Δ even and $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ for Δ odd; they gave a construction that showed this number is necessary. Andersen proved the conjecture for the case $\Delta = 3$ [1]. In this paper, we consider the case $\Delta = 4$.

Erdős and Nešetřil's construction for $\Delta = 4$ is shown in figure (1a). To form this graph, begin with a 5-cycle, then expand each vertex into two nonadjacent vertices who inherit all the neighbors of the original vertex. The graph has 20 edges, and contains no induced $2K_2$ (in fact, this is the largest graph that contains no induced $2K_2$ and

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has $\Delta = 4$ [2]). Hence, in a strong edge-coloring, every edge must receive its own color. The best upper bound previously known was 23 colors, proven by Horak [4]; we improve this upper bound to 22 colors.

We refer to the color classes as the integers from 1 to 22. A greedy coloring algorithm sequentially colors the edges, using the least color class that is not already prohibited from use on an edge at the time the edge is colored. By the *neighborhood* of an edge, we mean the edges which are distance at most 1 from the edge. Intuitively, this is the set of edges whose color could potentially restrict the color of that edge. We use the notation $N(e)$ to mean the edges in the neighborhood of e that are colored before edge e . Figure (1b) shows that the neighborhood of an edge has size at most 24. So for every edge order, we have $|N(e)| \leq 24$ for each edge. Thus, for every edge order, the greedy algorithm produces a strong edge-coloring that uses at most 25 colors. However, there is always some order of the edges for which the greedy algorithm uses exactly the minimum number of colors required. Our aim in this paper is to construct an order of the edges such that the greedy algorithm uses at most 22 colors. Throughout this paper, when we use the term coloring, we mean strong edge-coloring. Each connected component of G can be colored independently of other components, so we assume G is connected. We allow our graphs to include loops and multiple edges. We use δ to denote the minimum degree of the graph and we use $d(v)$ to denote the degree of vertex v . The girth of a graph is the length of the shortest cycle.

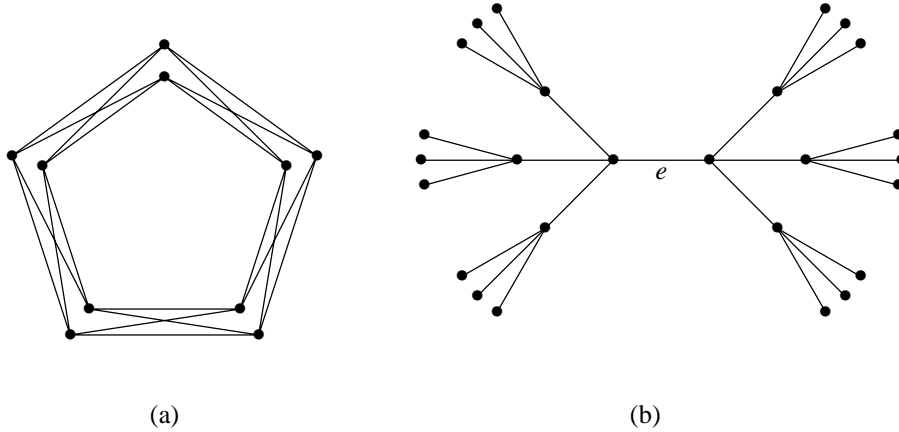


Figure 1: (a) Erdős and Nešetřil's construction for $\Delta = 4$. This graph requires 20 colors. (b) The largest possible neighborhood of an edge has size 24.

Let v be an arbitrary vertex of a graph G . Let $\text{dist}_v(v_1)$ denote the distance from vertex v_1 to v . Let *distance class* i be the set of vertices at distance i from vertex v . The distance class of an edge is the minimum of the distance classes of its vertices. We say that an edge order is *compatible* with vertex v if e_1 precedes e_2 in the order only when $\text{dist}_v(e_1) \geq \text{dist}_v(e_2)$. Intuitively, we color all the edges in distance class $i + 1$ (farther from v) before we color any edge in distance class i (nearer to v). Similarly, if

we specify a cycle C in the graph, we can define distance class i to be the set of vertices at distance i from cycle C . We say an edge order is compatible with C if e_1 precedes e_2 in the order only when $\text{dist}_C(e_1) \geq \text{dist}_C(e_2)$.

Lemma 1 *If G is a graph with maximum degree 4, then G has a strong edge-coloring that uses 21 colors except that it leaves uncolored those edges incident to a single vertex. If C is a cycle in G , then G has a strong edge-coloring that uses 21 colors except that it leaves uncolored the edges of C .*

Proof. We first consider the case of leaving uncolored only the edges incident to a single vertex. Let v be a vertex of G . Greedily color the edges in an order that is compatible with vertex v . Suppose we are coloring edge e , not incident to v . Let u be a vertex adjacent to an endpoint of e that is on a shortest path from e to v . Then none of the four edges incident to u has been colored, since each edge incident to u is in a lower distance class than e . Thus, $|N(e)| \leq 24 - 4 = 20$.

To prove the case of leaving uncolored only the edges of C , we color the edges in an order compatible with C . The argument above holds for every edge not incident to C . If e is incident to C and $|C| \geq 4$, then at least four edges in the neighborhood of e are edges of C ; so again $|N(e)| \leq 24 - 4 = 20$. If e is incident to C and $|C| = 3$, then by counting we see that the neighborhood of e has size at most 23. The three uncolored edges of C imply that $|N(e)| \leq 23 - 3 = 20$. \square

Lemma 1 shows that if a graph has maximum degree 4 we can color nearly all edges using at most 21 colors. In the rest of this paper, we show that we can always finish the edge-coloring using at most one additional color. Theorem 2 is the main result of this paper. We give the proof of the general case (4-regular and girth at least six) now, and defer the other cases (when the graph is not 4-regular or has small girth) to lemmas 4-10 in the remainder of the paper.

Theorem 2 *Any graph with maximum degree 4 has a strong edge-coloring with at most 22 colors.*

Lemma 3 *Any 4-regular graph with girth at least six has a strong edge-coloring with at most 22 colors.*

Proof. By Lemma 1, we choose an arbitrary vertex v , and greedily color all edges not incident to v , using at most 21 colors. Now we recolor edges e_1, e_2, e_3 , and e_4 (as shown in figure 2) using color 22. This allows us to greedily extend the coloring to the four edges incident to v . Edges e_1, e_2, e_3 , and e_4 can receive the same color since the girth of G is at least 6. \square

Lemma 3 proves Theorem 2 for 4-regular graphs with girth at least 6. To prove Theorem 2 for graphs that are not 4-regular and graphs with girth less than six, we use two ideas. In Section 2, we consider graphs that are not 4-regular and graphs with girth at most 3. In each case, we exploit local structure of the graph to give an edge order with $|N(e)| \leq 20$ for every edge in G . In Sections 3 and 4, we consider 4-regular graphs with girth 4 or 5. We find pairs of edges that can receive the same color. In this case, even though $|N(e)| > 21$, because not every edge in $N(e)$ receives a distinct color, we ensure that at most 22 colors are used.

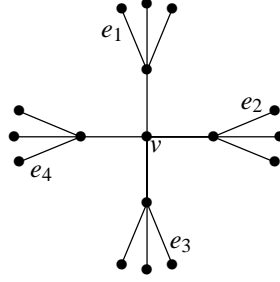


Figure 2: Vertex v has degree 4 and the girth of the graph is at least 6.

2 Graphs with $\delta < 4$ or girth at most 3

The three lemmas in this section are each proved using the same idea. We color nearly all edges as in Lemma 1. We show that due to the presence of a low degree vertex, a loop, a double edge, or a 3-cycle, it is possible to order the remaining uncolored edges so that a greedy coloring uses at most 21 colors.

Lemma 4 *Any graph with maximum degree 4 that has a vertex with degree at most 3 has a strong edge-coloring that uses 21 colors.*

Proof. We assume $d(v) = 3$ (if actually $d(v) < 3$, this only makes it easier to complete the coloring). Color the edges in an order that is compatible with vertex v . Let e_1, e_2, e_3 be the edges incident to vertex v . If the edges are ordered e_1, e_2, e_3 , we have $|N(e_1)| \leq 18, |N(e_2)| \leq 19$ and $|N(e_3)| \leq 20$, so there are colors for e_1, e_2 and e_3 . \square

Lemma 5 *A 4-regular graph with a loop or a double edge has a strong edge-coloring that uses 21 colors.*

Proof. If e is a loop incident to vertex v , we can greedily color the edges in an order compatible with vertex v (this is very similar to Lemma 4). So we can assume that the graph has a double edge.

Let v be one of the vertices incident to the double edge. Color the edges in an order that is compatible with vertex v . Let e_3, e_4 be the double edges and e_1, e_2 be the other edges incident to v . Then $|N(e_1)| \leq 17, |N(e_2)| \leq 18, |N(e_3)| \leq 16$ and $|N(e_4)| \leq 17$, so there are colors for e_1, e_2, e_3 , and e_4 . \square

Lemma 6 *A 4-regular graph with girth 3 has a strong edge-coloring that uses 21 colors.*

Proof. Let C be a 3-cycle in the graph. By Lemma 1 we greedily color all edges except the edges of C ; this uses at most 21 colors. An edge of a 3-cycle has a neighborhood of size at most 20, so each of the three uncolored edges satisfies $|N(e)| \leq 18$ and we can greedily finish the coloring. \square

3 4-regular graphs with girth four

Lemma 1 shows that we can color nearly all edges of the graph using 21 colors. Here we consider 4-regular graphs of girth four. We give an edge order such that the greedy coloring uses at most 22 colors; in some cases we precolor four edges prior to the greedy coloring. We use $A(e)$ to denote the set of colors available on edge e .

Lemma 7 *Any 4-regular graph with girth 4 has a strong edge-coloring that uses 22 colors.*

Proof. Let C be a 4-cycle, with the 4 edges labeled c_i ($1 \leq i \leq 4$) in clockwise order and the pair of edges not on the cycle and adjacent to c_i and c_{i-1} is labeled a_i and b_i (all subscripts are mod 4). We refer to the edges labeled by a_i and b_i as *incident edges*. By Lemma 1, we greedily color all edges except the edges of C and the 8 incident edges. This uses at most 21 colors. If two incident edges share an endpoint not on C , the two edges form an *adjacent pair*. The only possibility of an adjacent pair is if a_1 or b_1 shares an endpoint with a_3 or b_3 (or similarly if a_2 or b_2 shares an endpoint with a_4 or b_4). If the twelve uncolored edges contain at least two adjacent pairs, then we greedily color the incident edges. The neighborhood of each c_i has size at most 21, so $|A(c_i)| \geq 4$ for all i ; thus we can finish by greedily coloring the four edges of C .

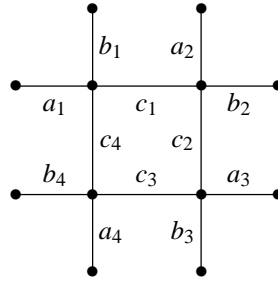


Figure 3: A 4-cycle in a 4-regular graph.

Suppose the uncolored edges contain exactly one adjacent pair. For example, suppose edges a_2 and a_4 share an endpoint. Call edges a_1, b_1, a_3 , and b_3 a *pack*. Consider the case when we can assign color 22 to two edges of the pack. Now we greedily color all edges except the edges of C . This uses at most 21 colors (Lemma 1). Each c_i has a neighborhood with size at most 22. Since color 22 is used twice in the neighborhood of each c_i , each c_i satisfies $|A(c_i)| \geq 4$. So we can greedily finish the coloring. Instead consider the case when no pair of edges in the pack can receive the same color. This implies the existence of edges between each pair of nonadjacent edges of the pack. Call these four additional edges *diagonal edges*. Observe (by counting) that the neighborhood of a diagonal edge has size at most 21. So we can color the diagonal edges last in the greedy coloring. Thus we greedily color all edges except the four edges of C and the four diagonal edges (this uses at most 21 colors). Now we color the four edges of

C (the four uncolored diagonal edges ensure there are enough colors available to color the edges of C). Lastly, we color the four diagonal edges.

Finally, suppose that the uncolored edges contain no adjacent pairs. In this case we will greedily color almost all edges of the graph (Lemma 1), but must do additional work beforehand to ensure that after greedily coloring most of the edges each c_i will satisfy $|A(c_i)| \geq 4$. As above, call edges a_1, b_1, a_3 , and b_3 a pack. Similarly, call edges a_2, b_2, a_4 , and b_4 a pack.

Consider the case when we can assign color 21 to two edges of one pack and assign color 22 to two edges of the other pack. We greedily color all edges but the four edges of C . Lemma 1 showed that a similar greedy coloring used at most 21 colors; however in Lemma 1 none of the edges were precolored. We adapt that argument to show that even in the presence of these four precolored edges a greedy coloring uses at most 22 colors. Lemma 1 argued there were at least four uncolored edges in the neighborhood of the edge being colored, so $|N(e)| \leq 20$. The same argument applies in this case except that possibly one of the edges that was uncolored in Lemma 1 is now colored. Hence $|N(e)| \leq 21$ (this follows from the fact that the four uncolored edges in Lemma 1 were incident to the same vertex and in the present situation at most one precolored edge is incident to each vertex). Hence, the greedy coloring uses at most 22 colors. The neighborhood of each c_i has size at most 23. Since colors 21 and 22 are each repeated in the neighborhood of each c_i , we see that each c_i satisfies $|A(c_i)| \geq 4$. So we can greedily finish the coloring.

Instead, consider the case when we can not assign color 21 to two edges of one pack and assign color 22 to two edges of the other pack. If no two edges in a pack can receive the same color, this implies the existence of edges between each pair of nonadjacent edges of the pack. These four diagonal edges each have a neighborhood with size at most 21. As we did above, we greedily color all edges except the four edges of C and the four diagonal edges. Now we color the four edges of C , and lastly, we color the four diagonal edges. \square

4 4-regular graphs with girth five

Here we consider 4-regular graphs with girth five. As in the case of girth four, we color nearly all the edges by Lemma 1. Intuitively, if there are enough different colors available to be used on the remaining uncolored edges, we should be able to complete this coloring by giving each uncolored edge its own color. However, if there are fewer different colors available than the number of uncolored edges, this approach is doomed to fail. Hall's Theorem formalizes this intuition. In the language of Hall's Theorem, we have m uncolored edges, and the set A_i denotes the colors available to use on edge i . For a proof of Hall's Theorem, we refer the reader to *Introduction to Graph Theory* [5].

Theorem 8 (Hall's Theorem) *There exists a system of distinct representatives for a family of sets A_1, A_2, \dots, A_m if and only if the union of any j of these sets contains at least j elements for all j from 1 to m .*

We define a *partial coloring* to be a strong edge-coloring except that some edges may be uncolored. Suppose that we have a partial coloring, with only the edge set T left uncolored. Let $A(e)$ be the set of colors available to color edge e . Then Hall's Theorem guarantees that if we are unable to complete the coloring by giving each edge its own color, there exists a set $S \subseteq T$ with $|S| > |\cup_{e \in S} A(e)|$. Define the *discrepancy*, $\text{disc}(S) = |S| - |\cup_{e \in S} A(e)|$.

Our idea is to color the set of edges with maximum discrepancy, then argue that this coloring can be extended to the remaining uncolored edges.

Lemma 9 *Let T be the set of uncolored edges in a partially colored graph. Let S be a subset of T with maximum discrepancy. Then a valid coloring for S can be extended to a valid coloring for all of T .*

Proof. Assume the claim is false. Since the coloring of S cannot be extended to $T \setminus S$, some set of edges $S' \subseteq (T \setminus S)$ has positive discrepancy (after coloring S). We show that $\text{disc}(S \cup S') > \text{disc}(S)$. Let R be the set of colors available to use on at least one edge of $(S \cup S')$. Let R_1 be the set of colors available to use on at least one edge of S . Let R_2 be the set of colors available to use on at least one edge of S' after the edges of S have been colored. Let $k = \text{disc}(S)$. Then $|S| = k + |R_1|$ and $|S'| \geq 1 + |R_2|$. Since S and S' are disjoint, we get

$$|S \cup S'| = |S| + |S'| \geq k + 1 + |R_1| + |R_2| > k + |R|.$$

The latter inequality holds since a color which is in $R \setminus R_1$ must be in R_2 and therefore we have $|R| = |R_1 \cup R_2| \leq |R_1| + |R_2|$. Hence

$$\text{disc}(S \cup S') = |S \cup S'| - |R| > k = \text{disc}(S)$$

This contradicts the maximality of $\text{disc}(S)$. Hence, any valid coloring of S can be extended to a valid coloring of T . \square

Lemma 10 *If G is a 4-regular graph with girth 5, then G has a strong edge-coloring that uses 22 colors.*

Proof. Let C be a 5-cycle, with the 5 edges labeled c_i ($1 \leq i \leq 5$) in clockwise order and the pair of edges not on the cycle and adjacent to c_i and c_{i-1} is labeled a_i and b_i (all subscripts are mod 5). We refer to the edges labeled by a_i and b_i as *incident edges*. Edge a_1 is at least distance 2 from at least one of edges a_3 and b_3 ; for if a_1 has edge e_1 to a_3 and edge e_2 to b_3 then we have the 4-cycle e_1, e_2, b_3, a_3 . Thus (by possibly renaming a_3 and b_3) we can assume there is no edge between edges a_1 and b_3 .

By repeating the same argument, we can assume there is no edge between the two edges of each of the following pairs: (a_1, b_3) , (a_3, b_5) , (a_5, b_2) , and (a_2, b_4) . Assign color 21 to edges b_1 and c_3 and assign color 22 to edges a_5 and b_2 . Greedily color all edges except the edges of C and the incident edges. This uses at most 22 colors.

There are 11 uncolored edges; if we can not assign a distinct color to each uncolored edge, then Hall's Theorem guarantees there exists a subset of the uncolored edges with positive discrepancy. Let S be a subset of the uncolored edges with maximum

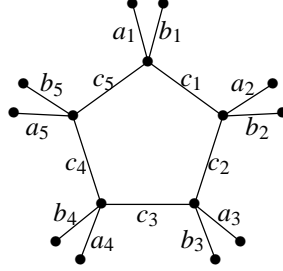


Figure 4: A 5-cycle in a 4-regular graph.

discrepancy. By counting the uncolored edges in the neighborhood of each edge, we observe that if e is an edge of C , then $|A(e)| \geq 8$ and if e is an incident edge then $|A(e)| \geq 5$. We can assume that S contains some edge of C , since otherwise we can greedily color S (Lemma 1), then extend the coloring to the remaining uncolored edges (Lemma 9). Since $\text{disc}(S) > 0$ and $|A(e)| \geq 8$ for each edge of C , we have $|S|$ is 9, 10, or 11.

Suppose $|S|$ is 9 or 10. Then since S is missing at most two uncolored edges, S contains at least one of the pair (a_1, b_3) , the pair (a_2, b_4) , and the pair (a_3, b_5) . Since each edge in the pair satisfies $|A(e)| \geq 5$ and $|\cup_{e \in S} A(e)| \leq 9$, some color is available for use on both edges of the pair. Assign the same color to both edges. Since the neighborhood of each uncolored incident edge, e , contains at least three uncolored edges of C , we have $|N(e)| \leq 24 - 3 = 21$; so we can greedily color the remaining uncolored incident edges. Now if S contains the pair (a_1, b_3) or the pair (a_3, b_5) then color the edges of the 5-cycle in the order c_2, c_4, c_5, c_1 ; if S contains the pair (a_2, b_4) then color the edges of the 5-cycle in the order c_2, c_4, c_1, c_5 .

Suppose $|S|$ is 11 and that no color is available on both edges of any of the pairs (a_1, b_3) , (a_2, b_4) , and (a_3, b_5) (otherwise the above argument holds). Assign the same color to c_1 and a_4 ; call it color x . Note that if $|A(c_1)| \geq 8$, $|A(a_4)| \geq 5$, and $|A(c_1) \cup A(a_4)| \leq |\cup_{e \in S} A(e)| \leq 10$, then $|A(c_1) \cap A(a_4)| \neq 0$. Before color x was assigned to c_1 and a_4 , it had been available on exactly one edge of each of the three pairs. Greedily color those three edges (none of the colors used on these three edges is color x). Now the three remaining uncolored incident edges each satisfy $|A(e)| \geq 3$, so we can greedily color them. Greedily color the three remaining edges in the order c_2, c_4, c_5 . \square

5 Conclusion

We note that it is straightforward to convert this proof to an algorithm that runs in linear time. We assume a data structure that stores all the relevant information about each vertex. Using breadth-first search, we can calculate the distance classes, as well as implement each lemma in linear time.

A natural question is whether it is possible to extend the ideas of this paper to larger Δ . The best bound we could hope for from the techniques of this paper is $2\Delta^2 - 3\Delta + 2$.

It is straightforward to prove an analog of Lemma 1 that gives a strong edge-coloring of G that uses $2\Delta^2 - 3\Delta + 1$ colors except that it leaves uncolored those edges incident to a single vertex (however, the author was unable to prove an analog to the “uncolored cycle” portion of Lemma 1). If G contains a loop, a double edge, or a vertex of degree less than Δ , then by the analog of Lemma 1 G has a strong edge-coloring that uses at most $2\Delta^2 - 3\Delta + 1$ colors. Using the ideas of Lemma 3, we see that if G is Δ -regular and has girth at least 6, then G has a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 2$ colors. Thus, to complete a proof for graphs with larger Δ , one must consider the case of regular graphs with girth 3, 4, or 5.

6 Acknowledgements

This paper draws heavily on ideas from a paper by Lars Andersen [1], in which he considers the case $\Delta(G) = 3$. The present author would like to thank him for that paper. Without it, this one would not have been written. The exposition of this paper has been greatly improved by critique from David Bunde and Erin Chambers.

References

- [1] L. D. Andersen. The strong chromatic index of a cubic graph is at most 10. *Topological, algebraical and combinatorial structures. Frolik's memorial volume. Discrete Math.*, 108(1-3):231–252, 1992.
- [2] F. R. K. Chung, A. Gyárfás, Zs. Tuza, and W. T. Trotter. The maximum number of edges in $2k_2$ -free graphs of bounded degree. *Discrete Math.*, 91(2):129–135, 1990.
- [3] R. J. Faudree, A. Gyárfás, R. H. Schelp, and Zs. Tuza. The strong chromatic index of graphs. *Ars Combinatoria*, 29B:205–211, 1990.
- [4] P. Horák. The strong chromatic index of graphs with maximum degree four. *Contemporary methods in graph theory*, pages 399–403, 1990.
- [5] D. B. West. *Introduction to Graph Theory*. Prentice Hall, second edition, 2001.
- [6] D. B. West. Strong edge-coloring. *Open Problems - Graph Theory and Combinatorics*. <http://www.math.uiuc.edu/~west/openp/strongedge.html>, 2003.